Michelson Interferometer

The interferometer experiment of Michelson and Morley played a crucial role in Einstein’s development of the Special Theory of Relativity.
Michelson Interferometer

A wavefront from the source is split into two equal amplitude components which are recombined at the detector.

Path lengths (1) $S \rightarrow M1 \rightarrow D$ and (2) $S \rightarrow M2 \rightarrow D$ are generally different. One mirror is moveable.
Michelson Interferometer

The operation of the device can be understood in terms of interference in a dielectric film. To see this we can conceptually “straighten out” the device by bringing the 2 arms into alignment.

The path length difference is $d$, the “mirror separation”. As in the film, circular Haidinger fringes are mapped out on the detector.
Michelson Interferometer

Michelson equivalent:

The optical path difference (OPD) between fields along paths (1) and (2) is:

\[ \text{OPD} = n(\overline{AB} + \overline{BC}) - n\overline{AD} = 2nd\cos\theta \]

giving the phase difference:

\[ k_0 \text{OPD} = 2k_0nd\cos\theta \]
The action of the beam splitter generally results in an additional phase difference of $\pi$ between the fields in paths (1) and (2). Field (2) undergoes an external reflection while field (1) undergoes an internal reflection at the beam splitter. Thus the overall phase difference is:

$$\delta = 2k_0 nd \cos \theta + \pi$$
**Michelson Interferometer**

Recombined beams at the detector give irradiance determined by the Two Beam Interference Formula (equal amplitude):

\[
I = 2I_0 \left(1 + \cos \delta \right)
\]

with

\[
\delta = 2\kappa_0 n d \cos \theta + \pi
\]

The condition for a dark fringe (irradiance minimum) is:

\[
\delta = (2m+1)\pi
\]

which can be written as:

\[
2nd \cos \theta = m\lambda_0
\]
Michelson Interferometer

The irradiance at the detector is usually written in terms of the quantity $\Delta$, which is the OPD between the two arms:

$$I = 2I_0 \left( 1 - \cos(k_0 \Delta) \right) = 4I_0 \sin^2\left(\frac{k_0 \Delta}{2}\right)$$

where

$$\Delta = 2n d \cos \theta = \text{OPD}$$
Michelson Interferometer

Behaviour of the fringes.

The dark fringe condition ($m^{th}$ order fringe):

$$2nd\cos\theta_m = m\lambda$$

⇒ for a given $m$, $\cos\theta_m$ decreases as $d$ increases
⇒ $\theta_m$ increases as $d$ increases.
⇒ Low order fringes move outward to higher angles with increasing $d$. 
Michelson Interferometer

Behaviour of the fringes.

The angular spacing of the fringes:

\[ \Delta \theta = \frac{\lambda}{2nd \sin \theta} \]

\( \Rightarrow \) for neighbouring fringes (\( \Delta m = 1 \)), \( \Delta \theta \) decreases as \( d \) increases.

\( \Rightarrow \) field of view becomes congested with more fringes as \( d \) increases.
Michelson Interferometer

Behaviour of the fringes.
**Michelson Interferometer**

*Behaviour of the fringes.*

Assuming 40 deg Field of View ie. $\theta < 20$ deg.

- $m = 0$  \hspace{1cm} $d = 0$

- Central bright fringe  \hspace{1cm} $d = \lambda/4$

- $m = 1$  \hspace{1cm} $d = \lambda/2$

- $m = 28 - 30$  \hspace{1cm} $d = 15\lambda$

- $m = 94 - 100$  \hspace{1cm} $d = 50\lambda$
Michelson Interferometer \[ m\lambda = 2d\cos\theta_m \]

<table>
<thead>
<tr>
<th>(d)</th>
<th>(m)</th>
<th>(\cos\theta_m)</th>
<th>(\theta) (degrees)</th>
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Michelson Interferometer

For misaligned mirrors, the Michelson interferometer is equivalent to the dielectric wedge geometry.

This gives the “Fringes of Equal Thickness” (“localized” at mirror M1). The central fringe is $\theta = 0$. 
Michelson Interferometer

For misaligned mirrors, the Michelson interferometer is equivalent to the dielectric wedge geometry.

This gives the “Fringes of Equal Thickness” (“localized” at mirror M1). The central fringe is $\theta = 0$. 

![Michelson Interferometer Image]
Michelson Interferometer: Applications

Measuring the Wavelength of Light

We set up the interferometer to monitor the central fringe ($\theta = 0$).

This can be done using a lens and an aperture in the lens focal plane to let only $\theta = 0$ light into the detector.
Michelson Interferometer: Applications

Measuring the Wavelength of Light

We set up the interferometer to monitor the central fringe ($\theta = 0$).

It can also be done using a telescope to visually align the $\theta = 0$ fringe with the telescope crosshairs. This will be the technique used in the lab.
Michelson Interferometer: Applications

The MI is set up to view the central fringe.

The dark fringe condition is:

\[ 2nd = m\lambda_0 \]

The order of the central fringe is:

\[ m = \frac{2nd}{\lambda_0} \]

If \( d \) is changed by \( \Delta d \) then \( m \) changes by \( \Delta m \):

\[ \Delta m = \frac{2n}{\lambda_0} \Delta d \]
Michelson Interferometer: Applications

\[ \Delta m = \frac{2\pi}{\lambda_o} \Delta d \]

An increase in \( d \) from \( d \rightarrow d + \Delta d \) gives an increase in \( m \) from \( m \rightarrow m + \Delta m \).

Thus, \( \Delta m \) fringes are counted at the detector as \( d \rightarrow d + \Delta d \).

By counting \( \Delta m \) fringes for known \( \Delta d \) we get:

\[ \lambda_o = 2\pi \frac{\Delta d}{\Delta m} \]

This can be a very accurate technique, especially when a large number of fringes are counted for an accurately measured mirror displacement \( \Delta d \).
Michelson Interferometer: Applications

The technique can be “inverted” to find $\Delta d$ in terms of wavelength. This is the basis of an atomic standard of length when a precisely known wavelength of an atomic spectral line is used.

$$\Delta d = \frac{d_0 \Delta \lambda}{2n}$$

Interferometer used as length standard at the National Metrology Institute of Japan.
Michelson Interferometer: Applications

Refractive Index of a Gas

In the arrangement shown, a gas cell is placed in one arm of the MI and initially evacuated \((n_{\text{vac}} \equiv 1)\). Gas is then leaked into the cell until a desired pressure is reached. The corresponding refractive index is \(n_g\) (at that pressure).

As gas leaks into the cell, the order of the central fringe changes.
Michelson Interferometer: Applications

Assume the central fringe is initially dark (the MI can be set up this way). The (central) dark fringe condition is:

\[ 2nd = m \lambda_0 \Rightarrow m = \frac{2nd}{\lambda_0} \]

As gas pressure increases, \( n \) will change in the cell and there will be a corresponding change in \( m \):

\[ \Delta m = \frac{2}{\lambda_0} \Delta(nd) \]

\( \Delta m \) is the number of central fringes counted with the change in OPD of \( \Delta(nd) \).
Michelson Interferometer: Applications

The change in OPD:

$$\Delta m = \frac{2}{\lambda_0} \Delta(n d)$$

$$\Delta(n d) = n_g L - n_{vac} L = (n_g - 1) L$$
Michelson Interferometer: Applications

Refractive Index of a Glass Plate

Similar type of measurement to the gas cell experiment. Here, we begin by setting up the condition of Zero OPD. How? We look for the presence of “white light fringes” (more later).
Michelson Interferometer: Applications

Refractive Index of a Glass Plate

Similar type of measurement to the gas cell experiment. Here, we begin by setting up the condition of Zero OPD. How? We look for the presence of “white light fringes” (more later).

Then insert the glass plate. The change in OPD with the glass plate inserted:

$$\Delta(nd) = n_p t - n_{\text{air}} t \approx (n_p - 1) t$$
Michelson Interferometer: Applications

Refractive Index of a Glass Plate

Now, with the glass plate inserted, the mirror separation is adjusted to reestablish the “white light fringe” observation condition. If the change in mirror separation required to re-capture the white light fringes is $\Delta d$, then:

$$\Delta d = (n_p - 1) t$$

and the index of refraction can be determined if the thickness of the plate is known.
The MI can be used to measure the wavelength separation between ("resolve") two closely spaced spectral lines (eg the Na"Doublet").

\[
\Delta \lambda_0 = 0.597 \text{ nm} \\
\lambda_0 = 589 \text{ nm}
\]

\[
3^2P_{3/2} \quad 3^2P_{1/2} \\
\quad \quad 589.593 \text{ nm} \quad 588.996 \text{ nm} \\
\quad \quad 3^2S_{1/2}
\]
Michelson Interferometer: Applications

Each wavelength component gives rise to its own fringe pattern (with different fringe spacing). At some mirror separations, \(d\), dark fringes from \(\lambda_01\) will overlap bright fringes from \(\lambda_02\) and the combined fringe pattern will “wash out” ie no fringes will be visible.

The irradiance at the detector due to each wavelength component:

\[
\begin{align*}
\lambda_{01}: \quad I_1 &= 2I_{01}(1 - \cos(k_0 \Delta)) = 4I_{01} \sin^2\left(\frac{k_0 \Delta}{2}\right) \\
\lambda_{02}: \quad I_2 &= 2I_{02}(1 - \cos(k_2 \Delta)) = 4I_{02} \sin^2\left(\frac{k_2 \Delta}{2}\right)
\end{align*}
\]

with \(\Delta = 2\pi d \cos \theta\)

The total irradiance at the detector:

\[
I = I_1 + I_2
\]
Michelson Interferometer: Applications

Assuming $I_{01} \equiv I_{02} \equiv \frac{1}{2} I_0$ (where $I_0$ is the total irradiance from both wavelength components in each arm) the irradiance at the detector becomes:

\[
I = 2I_0 \left\{ \sin^2 \left( \frac{k_{01} \Delta}{2} \right) + \sin^2 \left( \frac{k_{02} \Delta}{2} \right) \right\}
\]

\[
= 2I_0 \left\{ 1 - \cos \left[ \frac{(k_{01} + k_{02}) \Delta}{2} \right] \cos \left[ \frac{(k_{01} - k_{02}) \Delta}{2} \right] \right\}
\]
Michelson Interferometer: Applications

We rewrite this as:

\[ I = 2I_0 (1 - \cos Ax \cos Bx) \]

\[ \frac{I}{I_0} \]

\[ A = k_{01} + k_{02} \]
\[ B = k_{01} - k_{02} \]
\[ x = n d \cos \theta \]

The \( \cos Ax \) term is rapidly varying (generating the fringes) while the \( \cos Bx \) term is slowly varying.
Michelson Interferometer: Applications

We rewrite this as:

\[ I = 2I_0 (1 - \cos Ax \cos Bx) \]

\[ A = k_{01} + k_{02} \]
\[ B = k_{01} - k_{02} \]
\[ \chi = n d \cos \Theta \]

The \( \cos Ax \) term is rapidly varying (generating the fringes) while the \( \cos Bx \) term is slowly varying (the fringe “envelope”).
Michelson Interferometer: Applications

We rewrite this as:

\[ I = 2I_0 (1 - \cos Ax \cos Bx) \]

\[ \frac{A}{k_0} = k_{01} + k_{02} \]
\[ B = k_{01} - k_{02} \]
\[ \chi = nd \cos \theta \]

The \( \cos Ax \) term is rapidly varying (generating the fringes) while the \( \cos Bx \) term is slowly varying (the fringe “envelope”).

The fringes vanish when \( \cos Bx = 0 \)

ie

\[ Bx = (2m+1)\pi / 2 \]
Michelson Interferometer: Applications

Thus the fringes vanish ("wash out") when:

\[ \cos B_x = 0 \]

\[ \Rightarrow B_x = (2m+1) \frac{\pi}{2} \]

\[ \Rightarrow (k_{o1} - k_{o2}) n d_m \cos \theta = (2m+1) \frac{\pi}{2} \]

We can calculate the difference in wavenumber \( \Delta k_o \) of the 2 spectral lines if we measure the mirror separation \( d_m \) corresponding to the \( m^{th} \) order in which the fringes disappear:

\[ \Delta k_o = \frac{(2m+1) \pi}{2 n d_m \cos \theta} \]

\[ \Delta k_o = k_{o1} - k_{o2} \]
Michelson Interferometer: Applications

It’s sometimes more meaningful to find the difference in wavelength $\Delta \lambda_0$ rather than difference in wavenumber $\Delta k_0$ of the 2 spectral lines.

We can relate the two differences by:

$$\Delta \lambda_0 = \frac{\partial \lambda_0}{\partial k_0} \Delta k_0$$

$$\lambda_0 = \frac{2\pi}{k_0} \Rightarrow \Delta \lambda_0 = -\frac{\lambda_0^2}{2\pi} \Delta k_0$$

Where $\lambda_0$ is taken to be the mean value:

$$\lambda_0 = \frac{1}{2} (\lambda_{01} + \lambda_{02})$$

So:

$$|\Delta \lambda_0| = \lambda_0^2 \frac{(2m+1)}{4n d_m \cos \theta}$$

or

$$|\Delta \lambda_0| = (2m+1) \frac{\lambda_0^2}{4d_m}$$

for normal viewing ($\theta = 0$).
A better technique (as you will use in the lab) involves measuring the change in mirror separation $\Delta d_m$ required to go from one fringe disappearance ($m^{th}$ order) to the next ($m+1)^{th}$ order. For this measurement it’s unnecessary to know the order, $m$, to determine the wavelength difference, $\Delta \lambda_0$. 

\[
\Delta d_m \\
\Delta \lambda_0
\]
Michelson Interferometer: Applications

The change in mirror separation $\Delta d_m$ required to go from one fringe disappearance ($m^{th}$ order) to the next ($(m+1)^{th}$ order) gives $\Delta \lambda_0$:

$$d_m = (2m+1) \frac{\lambda_0^2}{4 \Delta \lambda_0}$$

$$d_{m+1} = (2m+3) \frac{\lambda_0^2}{4 \Delta \lambda_0}$$

$$\Rightarrow \Delta d_m = d_{m+1} - d_m = \frac{\lambda_0^2}{2 \Delta \lambda_0}$$

$$\Rightarrow \Delta \lambda_0 = \frac{\lambda_0^2}{2 \Delta d_m}$$
Michelson Interferometer: Applications

White Light Fringes

For the monochromatic (single wavelength) source:

\[ I(\Delta) = 2I_0 (1 - \cos(k_0 \Delta)) \]
Michelson Interferometer: Applications

For the doublet (two wavelength) source:

\[ I = \sum_{\hat{c}} I_{\hat{c}}(\Delta) = 2I_0 \sum_{\hat{c}} (1 - \cos(k_{0i} \Delta)) \]
Michelson Interferometer: Applications

For the white light source:

\[ I = 2 I_0 \int_{-\infty}^{\infty} dk \, f(k) \left( 1 - \cos(k\Delta) \right) \]

with \( \int_{-\infty}^{\infty} dk \, f(k) = 1 \)

Spectrum

Fringe Pattern: \( I(\Delta) \)
Michelson Interferometer: Applications

An approximation to a white light spectrum: “Model Spectrum”

Here: \[ I(\Delta) = \frac{2I_0}{(k_{02} - k_{01})} \int_{k_{01}}^{k_{02}} dk \left( 1 - \cos(k\Delta) \right) \]

\[ I_{\text{source}} = \int dk \frac{2I_0}{(k_{02} - k_{01})} = 2I_0 \]

\( I_{\text{source}} \) is the total (integrated over all wavelengths) intensity from the source. \( I_0 \) is the integrated intensity in one arm of the MI.
Michelson Interferometer: Applications

By evaluating the integral

\[ I(\Delta) = \frac{2I_o}{(k_{o2} - k_{o1})} \int_{k_{o1}}^{k_{o2}} dk \left(1 - \cos(k\Delta)\right) \]

we can show that (details follow):

\[ I(\Delta) = 2I_o \left\{ 1 - \sin \left[ \frac{(k_{o2} - k_{o1})\Delta}{2} \right] \cos \left[ \frac{(k_{o2} + k_{o1})\Delta}{2} \right] \right\} \]
Michelson Interferometer: Applications

or:

\[ I(\Delta) = 2I_0 \left\{ 1 - \text{sinc} \left[ \frac{(k_{02} - k_{01})\Delta}{2} \right] \cos \left[ \frac{(k_{02} + k_{01})\Delta}{2} \right] \right\} \]

where the “sinc” function is defined by \( \text{sinc}(x) = \frac{\sin x}{x} \)
Michelson Interferometer: Applications

For the white light “model spectrum” source:

\[ I(\Delta) = 2I_0 \left\{ 1 - \text{sinc} \left[ \frac{(k_{02} - k_{01}) \Delta}{2} \right] \cos \left[ \frac{(k_{02} + k_{01}) \Delta}{2} \right] \right\} \]

Fringe pattern looks somewhat like the real deal!
Michelson Interferometer: Applications

Details: Evaluation of the model white light fringe pattern.

\[ I(\Delta) = \frac{2I_0}{(k_{02} - k_{01})} \int_{k_{01}}^{k_{02}} dk (1 - \cos(k\Delta)) \]

\[ = \frac{2I_0}{(k_{02} - k_{01})} \left[ k - \frac{\sin(k\Delta)}{\Delta} \right]_{k_{01}}^{k_{02}} \]

\[ = \frac{2I_0}{(k_{02} - k_{01})} \left( (k_{02} - k_{01}) - \frac{\sin(k_{02}\Delta) - \sin(k_{01}\Delta)}{\Delta} \right) \]
Michelson Interferometer: Applications

Details (cont.)

\[ \sin u - \sin v = 2 \sin \left( \frac{u-v}{2} \right) \cos \left( \frac{u+v}{2} \right) \]

\[ I(\Delta) = \frac{2I_0}{(k_{02}-k_{01})} \left( k_{02}-k_{01} \right) - \frac{2}{\Delta} \sin \left( \frac{(k_{02}-k_{01})\Delta}{2} \right) \cos \left( \frac{(k_{02}+k_{01})\Delta}{2} \right) \]

\[ = 2I_0 \left\{ 1 - \frac{\sin \left[ \frac{(k_{02}-k_{01})\Delta}{2} \right]}{\left[ \frac{(k_{02}-k_{01})\Delta}{2} \right]} \cos \left[ \frac{(k_{02}+k_{01})\Delta}{2} \right] \right\} \]

\[ = 2I_0 \left\{ 1 - \text{sinc} \left[ \frac{(k_{02}-k_{01})\Delta}{2} \right] \cos \left[ \frac{(k_{02}+k_{01})\Delta}{2} \right] \right\} \]
Optical coherence tomography is a recently developed, noninvasive technique for imaging subsurface tissue structure with micrometer-scale ($10^{-6}$ m) resolution. Depths of 1–2 mm can be imaged in turbid tissues such as skin or arteries; greater depths are possible in transparent tissues such as the eye.
Michelson Interferometer: Applications

Optical Coherence Tomography

SLD: superluminescent Diode (white light source)  REF: (reference) mirror
SMP: sample  BS: beamsplitter  CO-CAM: camera objective - camera

The device is a Michelson interferometer used with white light!
Michelson Interferometer: Applications

Optical Coherence Tomography

Light that has been back-reflected from the tissue sample (index-of-refraction mismatches) and light from the reference arm recombine. Because the source is a “broadband” (white light) source, only light which has traveled very close to the same optical path length in the reference and tissue arms will generate interference fringes. The fringes are detected at the camera. By changing the length of the reference arm, reflection sites at various depths in the tissue can be sampled.
Michelson Interferometer: Applications

Variations of the Michelson Interferometer.
Twyman Green interferometer uses a point source and collimating lens as opposed to the extended source in the Michelson. As shown it is configured to test a lens.
Michelson Interferometer: Applications

Twyman Green variation:

Fringe patterns for an aberrated lens:

Spherical Aberration  Coma  Astigmatism
Michelson Interferometer: Applications

Mach Zehnder Interferometer (Fusion Plasma Diagnostics)
Michelson Interferometer: Applications

Linnik Interference Microscope.

Used for mapping surface topography.
Michelson Interferometer: Applications

Linnik Interference Microscope.

linnik.gif
Michelson Interferometer: Applications

LIGO: Laser Interferometer Gravitational Wave Observatory

What Will LIGO Observe?

Gravitational waves triggered by cosmic events should cause specific displacements resulting in unique interference patterns.
Michelson Interferometer: Applications

LIGO: Laser Interferometer Gravitational Wave Observatory

LIGO CalTech prototype (40m arms).
Michelson Interferometer: Applications

LIGO: Laser Interferometer Gravitational Wave Observatory

LIGO Hanford Observatory, Washington State.